Topology — Worksheet 4

Qualifying Exam Prep Seminar 2020

1. Two continuous maps $f, g : X \to Y$ are **homotopic** if there is a continuous map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ and we call H a **homotopy** from f to g.

   (a) Let $\mathrm{Hom}(X, Y)$ be the set of continuous maps from X to Y. Prove that homotopy is an equivalence relation on this set.

   (b) Suppose $f, g : X \to Y$ are homotopic and $h, k : Y \to Z$ are homotopic. Prove that $h \circ f$ is homotopic to $k \circ g$. In particular, if $f\colon X \to X$ is homotopic to the identity map, the $f \circ g$ and $g \circ f$ are both homotopic to g.

   Congratulations! You've just shown that there is a category whose objects are topological spaces and morphisms are equivalence classes of continuous maps.

   (c) What is a common vocab word you would use to describe a representative of an isomorphism in this category?

2. Here are some exercises we gave my 145A students on quizes last winter. For each problem state and prove a general claim about metrics spaces.

   (a) Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a continuous function. Prove that the function
   $$d'(x, y) := d(x, y) + |f(x) - f(y)|$$
   also defines a metric on X.

   (b) Let $r(x, y) = |x^2 - y^2|$ where x and y are real numbers. Prove that r defines a metric on the set $[0, \infty)$ of nonnegative real numbers but does not define a metric on the set $\mathbb{R}$ of all real numbers.