Topology — Worksheet 2

Qualifying Exam Prep Seminar 2020

1. Let's brush up on some vector calculus. When you teach 10A, you tell people that, given a function $f\colon \mathbb{R}^3 \to \mathbb{R}$ and a vector $\mathbf{v}$, the **directional derivative** of f in the $\mathbf{v}$ direction is given by:

$$\mathbf{v} \cdot \nabla f$$

I should mention that one would usually have to say that $\mathbf{v}$ is a unit vector or otherwise we normalize it, but it more helpful for these exercises to think of directional derivatives also having magnitude.

   (a) Pick your favorite point $p = (x_1, y_1, z_1) \in \mathbb{R}^3$. Like, literally pick an actual triple of real numbers. You'll be doing math with them so don't pick anything too crazy. Given each of the following smooth functions $f\colon \mathbb{R}^3 \to \mathbb{R}$, compute the **directional derivative** of f in the $(1,1,1)$ direction and evaluate at p.

$$f(x,y,z) = x^2 + y^2 + z^2 \qquad f(x,y,z) = 2x^2 + 2y^2 + 2z^2$$

$$f(x,y,z) = \sin(xy) \qquad f(x,y,z) = \cos(yz) \qquad f(x,y,z) = \sin(xy) + \cos(yz)$$

   (b) After fixing a point p, I could replace $(1,1,1)$ with a lot of things. What's a nice description of the space of directions in which I can take a derivative of $f : \mathbb{R}^3 \to \mathbb{R}$? Your answer should be a vector space; can you think of a basis for this vector space?

   (c) Verify that taking the derivative in blah direction and evaluating at blah gives a functional $\mathscr{C}^\infty(\mathbb{R}^3) \to \mathbb{R}$ which is $\mathbb{R}$-linear and satisfies the product rule.

   (d) Recall the definitions of a **derivation** and the **(algebraic) tangent space** at a point $p \in M$.

2. (a) Consider the following maps $X : \mathbb{R}^2 \to \mathbb{R}^2$. Sketch each by drawing $X(p)$ as a vector based at p.

   - $X(x,y) = (1,0)$
   - $X(x,y) = (x,0)$
   - $X(x,y) = (x,y)$

   (b) In the previous problem, we formalized what we mean by "vectors based at a point." Naively, we can think of this map as

$$X : \mathbb{R}^2 \to T_p\mathbb{R}^2\,,$$

   however, the p in the codomain would depend on the point inputted into X. To remedy this, we consider it instead as a map

$$X : \mathbb{R}^2 \to \bigsqcup_{p\in\mathbb{R}^2} T_p\mathbb{R}^2\,.$$

   What extra condition must X satisfy to ensure that the vector is based at the correct point?

   (c) Recall the definitions of **tangent bundle** and **vector field**.