# WS 1 #2:

Given a simplicial complex $K$ and a simplicial sub complex $L$ of $K$, we get the following Long exact sequence of relative homology:

$$\cdots \longrightarrow H_n(L) \xrightarrow{(1)} H_n(K) \xrightarrow{(2)} H_n(K,L) \longrightarrow H_{n-1}(L) \to \cdots$$

Describe the maps (1) and (2).

Consider the inclusion map $\iota : L \hookrightarrow K$ and the induced map on chain complexes $\iota_*^n : C_n(L) \to C_n(K)$. $\iota_*^n$ is injective since it is injective on a basis of $C_n(L)$. So we get the following short exact sequence:

$$0 \to C_n(L) \xrightarrow{\iota_*^n} C_n(K) \xrightarrow{\pi^n} C_n(K,L) \to 0$$

Along with the rest of the chain complex, we get

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_{n+1}(L) \arrow[r, "\iota_*^{n+1}"] \arrow[d, "\partial_{n+1}"] & C_{n+1}(K) \arrow[r, "\pi^{n+1}"] \arrow[d, "\partial_{n+1}"] & C_{n+1}(K,L) \arrow[r] \arrow[d, "\partial_{n+1}"] & 0 \\
0 \arrow[r] & C_n(L) \arrow[r, "\iota_*^n"] \arrow[d, "\partial_n"] & C_n(K) \arrow[r, "\pi^n"] \arrow[d, "\partial_n"] & C_n(K,L) \arrow[r] \arrow[d, "\partial_n"] & 0 \\
0 \arrow[r] & C_{n-1}(L) \arrow[r, "\iota_*^{n-1}"] \arrow[d] & C_{n-1}(K) \arrow[r, "\pi^{n-1}"] \arrow[d] & C_{n-1}(K,L) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Each column here gives us a homology and the rows induce the maps (1) and (2). Let's flesh out some of those details.

## WS1 #2 (cont'd):

The map (1) is the map induced by the inclusion $\imath : L \hookrightarrow K$.

More Precisely, $\imath_* : H_n(L) \to H_n(K) : [x] \longmapsto [\imath_*^n(x)]$. We can similarly define the map (2) by $\pi_* : H_n(K) \to H_n(K,L) : [x] \mapsto [\pi^n x]$. However, there remains some questions of well-definedness for both of these maps. Starting with $\imath_*$:

① Given $[x] \in H_n(L)$, then $x \in \ker(\partial_n) \subseteq C_n(L)$. Why is $\imath_*^n(x) \in \ker(\partial_n) \subseteq C_n(K)$?

② If $[x] = [y] \in H_n(L)$, then why does $[\imath_*^n(x)] = [\imath_*^n(y)] \in H_n(K)$?

To answer ①, use the commutativity of the diagram above. Specifically,

$$\imath_*^{n-1} \circ \partial_n = \partial_n \circ \imath_*^n. \text{ Since } x \in \ker(\partial_n),\ \partial_n(\imath_*^n(x)) = \imath_*^{n-1}(\partial_n(x)) = 0,$$

so $\imath_*^n(x) \in \ker(\partial_n)$.

For ②, note $[x-y] = [0] \in H_n(L)$, so $x - y \in \operatorname{Im}(\partial_{n+1}) \subseteq C_n(L)$. Let $z \in C_{n+1}(L)$ be such that $\partial_{n+1}(z) = x - y$. The by the commutativity, we get: $\imath_*^n(x-y) = \imath_*^n(\partial_{n+1}(z)) = \partial_{n+1}(\imath_*^{n+1}(z)) \in \operatorname{Im}(\partial_{n+1}) \subseteq C_n(K)$, so $[\imath_*^n(x-y)] = [0]$, thus $[\imath_*^n(x)] = [\imath_*^n(y)]$.

This is a general phenomenon. In general, if you have maps $\varphi_n : C_n(K_1) \to C_n(K_2)$ such that the following commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(K_1) \arrow[r, "\partial_{n+1}"] \arrow[d, "\varphi_{n+1}"] & C_n(K_1) \arrow[r, "\partial_n"] \arrow[d, "\varphi_n"] & C_{n-1}(K_1) \arrow[r] \arrow[d, "\varphi_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(K_2) \arrow[r, "\partial_{n+1}"] & C_n(K_2) \arrow[r, "\partial_n"'] & C_{n-1}(K_2) \arrow[r] & \cdots
\end{tikzcd}$$

We get induced maps $\varphi_* : H_n(K_1) \to H_n(K_2)$.