Topology — Mock Exam 3

Topology Qualifying Prep Seminar 2020

There are three parts in this exam and each part has three problems. You should complete two and only two problems of your choice in each part. Each problem is worth 10 points.

Support each answer with a complete argument. State completely any definitions and basic theorems that you use.

This is a closed book test. You may use only the test, something with which to write, and blank paper. All other material is prohibited. Write each of your solutions on separate sheets of paper. Write your student ID number on every sheet you use.

The time for this exam is 3 hours.

PART I

1. Let $X := C^0(\mathbb{R}, \mathbb{R})$ be the set of continuous functions from $\mathbb{R}$ to itself. Given a continuous function g in this set and a continuous function
$$\varepsilon \colon \mathbb{R} \to (0, \infty)$$
define
$$U_\varepsilon(g) := \{h \in C^0(\mathbb{R}, \mathbb{R}) \;:\; |h(x) - g(x)| < \varepsilon(x) \text{ for all } x \in \mathbb{R}\}.$$
   (a) Show that the collection of all such $U_\varepsilon(g)$ form a basis for a topology on X, called the **fine C^0 topology**.
   (b) Show that the fine C^0 topology is not first countable, and explain why it is also not metrizable.

2. Let $T^2 = S^1 \times S^1$ be the torus with its usual topology. Let $X \subseteq T^2$ be the subset $S^1 \times \{*\}$, where $* \in S^1$ is any point.
   (a) Is X a retract of T^2? Prove your answer.
   (b) Is X a deformation retract of T^2? Prove your answer.

3. A topological space X is **irreducible** if whenever X is decomposed as
$$X = A \cup B$$
with A and B closed subsets, then either $A = X$ or $B = X$.

   Prove that an irreducible Hausdorff space consists of at most one point.

PART II

4. Compute the fundamental group of S^2 with n points removed.

5. Suppose that any map $X \to X$ has a fixed point, and that $A \subseteq X$ is a retract of X. Prove that any map $A \to A$ has a fixed point.

6. Let X be the space obtained by gluing a copy of S^1 and a copy of S^2 along a point. Compute $\pi_1(X)$ and describe the (uniquely defined) n-fold cover $\tilde{X}$ of X.

PART III

7. Show that a connected smooth manifold is **smoothly homogeneous** in the sense that given any $p, q \in M$ there is a diffeomorphism
$$\Phi \colon M \to M$$
so that $\Phi(p) = q$.

   You may assume that any two points $p, q \in M$ can be connected by a smooth embedded curve, γ, and that a vector field with compact support is "complete" in the sense that it generates a global flow.

8. Let $M_n(\mathbb{C})$ be the space of $n \times n$ matrices with complex entries.

   (a) Show that $M_n(\mathbb{C})$ can be identified with a real vector space, and determine its (real) dimension.

   (b) Denote by $GL_n(\mathbb{C})$ the subset of invertible matrices. Prove that it is an open subset of $M_n(\mathbb{C})$.

   (c) On $M_n(\mathbb{C})$ we can naturally define the map $\det\colon M_n(\mathbb{C}) \to \mathbb{C}$. Let $SL_n(\mathbb{C}) = \det^{-1}(1)$; show it is a smooth submanifold of $GL_n(\mathbb{C})$ and compute its (real) dimension.

9. Show that the subset of $\mathbb{R}^3$ defined by the equation

$$(1 - z^2)(x^2 + y^2) = 1$$

   is a smooth manifold.