Topology — Mock Exam 2

Topology Qualifying Prep Seminar 2020

There are three parts in this exam and each part has three problems. You should complete two and only two problems of your choice in each part. Each problem is worth 10 points.

Support each answer with a complete argument. State completely any definitions and basic theorems that you use.

This is a closed book test. You may use only the test, something with which to write, and blank paper. All other material is prohibited. Write each of your solutions on separate sheets of paper. Write your student ID number on every sheet you use.

The time for this exam is 3 hours.

PART I

1. Let X be a compact metric space and

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$$

be an infinite sequence of closed non-empty subsets of X. Prove that

$$\bigcap_{i=1}^{\infty} C_i \neq \emptyset$$

2. Suppose that (X, d) is a metric space. Show that the topology on X determined by d is the coarsest topology on X such that the metric $d\colon X \times X \to \mathbb{R}$ is continuous.

3. Show that $\mathbb{Q}$ as a subset of $\mathbb{R}$ is not locally compact.

PART II

4. The **suspension** of a space X is the quotient space of $X \times [0, 1]$ obtained by collapsing each of $X \times \{0\}$ and $X \times \{1\}$ to points.

   (a) Compute the fundamental group of the suspension of $\mathbb{RP}^n$.

   (b) Compute the homology groups of the suspension of $\mathbb{RP}^n$.

5. Let $p\colon X \to B$ be a covering map and G its group of covering transformations. Prove there is a homeomorphism $k\colon X/G \to B$ such that $p = k \circ \pi$ where $\pi\colon X \to X/G$ is the projection.

6. Let X be a path-connected space. Prove that $H_0(X; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$.

PART III

7. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F(x, y) = (x + y^2, x^2).$$

Denoting by u and v the Cartesian coordinates of the target, determine $F^*(v\,du + dv)$

8. A **Riemannian metric** on a smooth manifold M is an assignment of an inner product on each tangent space:

$$g_p\colon T_pM \times T_pM \to \mathbb{R}$$

which varies smoothly with respect to p, that is, given any smooth vector fields X and Y, the map

$$M \to \mathbb{R} : p \mapsto g_p(X_p, Y_p)$$

is smooth. Prove that every smooth manifold admits a Riemannian metric.

9. Let $p\colon M \to N$ be a covering map between topological manifolds. Show that if N is a smooth manifold, then M admits a unique smooth structure such that p is a smooth submersion and immersion.