Topology — Mock Exam 1

Topology Qualifying Prep Seminar 2020

There are three parts in this exam and each part has three problems. You should complete two and only two problems of your choice in each part. Each problem is worth 10 points.

Support each answer with a complete argument. State completely any definitions and basic theorems that you use.

This is a closed book test. You may use only the test, something with which to write, and blank paper. All other material is prohibited. Write each of your solutions on separate sheets of paper. Write your student ID number on every sheet you use.

The time for this exam is 3 hours.

PART I

1. Let $p \in S^1$.

   (a) Show that there exists a homeomorphism $f\colon S^1 \backslash \{p\} \to \mathbb{R}$.

   (b) Prove that no homeomorphism $f\colon S^1 \backslash \{p\} \to \mathbb{R}$ can be extended to a continuous $\hat{f}\colon S^1 \to \mathbb{R}$.

   (c) Prove that no continuous map $g\colon S^1 \to \mathbb{R}$ can be injective.

2. Let d be the standard Euclidean metric on $\mathbb{R}^n$ and let $A \subseteq \mathbb{R}^n$ be closed. Show that for each $x \in \mathbb{R}^n$, there is a point $y \in A$ of minimum distance from x.

3. Prove that any compact metric space has a countable dense subspace.

PART II

4. Let X be the union of the unit sphere $S^2 \subseteq \mathbb{R}^3$ and any three distinct diameters of S^2.

   (a) Compute $\pi_1(X)$.

   (b) Compute $H_n(X)$ for all n.

5. Let F be the free group on two generators $F = \langle a, b \rangle$. Consider the homomorphism $f\colon F \to \mathbb{Z}/3\mathbb{Z}$ defined by $a \mapsto [2]$ and $b \mapsto [1]$. Give a set of free generators for the kernel of f.

6. Let X be a topological manifold, i.e. a space which is second countable, locally Euclidean, and Hausdorff. Compute

$$H_*(X, X - \{x\})$$

   for all $x \in X$.

PART III

7. Let M be a manifold with boundary, and let

$$H^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \ : \ x_n \geqslant 0\} \, ,$$

$$\partial H^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \ : \ x_n = 0\},$$

$$\mathrm{Int} H^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \ : \ x_n > 0\}], .$$

   Recall that

$$\partial M := \{p \in M \ : \ \exists (U, \varphi) \text{ with } \varphi(p) \in \partial H^n\}$$

$$\mathrm{Int} M := \{p \in M \ : \ \exists (U, \varphi) \text{ with } \varphi(p) \in \mathrm{Int} H^n\}$$

   Prove that $\partial M \cap \mathrm{Int} M = \emptyset$.

8. Prove that every Lie group is parallelizable.

9. Recall that $\mathbb{RP}^n = \mathbb{R}^{n+1} - \{0\}/\mathbb{R}$ and a point in $\mathbb{RP}^n$ can be written in **homogenous coordinates**, i.e. an equivalence class $[x_0, \ldots, x_n]$ of a point $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} - \{0\}$. Consider the map $\Phi : \mathbb{RP}^2 \to \mathbb{RP}^2$ defined as:

$$[x, y, z] \mapsto \left[\frac{2x}{x^2 + y^2 + z^2}, \frac{y}{x^2 + y^2 + z^2}, \frac{2z}{x^2 + y^2 + z^2} \right].$$

Determine the rank of $\Phi_\star$ at $[0, 1, 0] \in \mathbb{RP}^2$.