Topology — Homework 2

Qualifying Exam Prep Seminar 2020

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1. Recall $S^1 := \{z \in \mathbb{C} \,:\, |z| = 1\}$ and define $g, h\colon S^1 \to S^1$ by $g(z) = z^n$ and $h(z) = z^{-n}$. Compute the induced homomorphisms g_* and h_* of the infinite cyclic group $\pi_1(S^1, b_0)$ into itself.
   [Source: Fall 2019 205A Homework Set 6 Question 3]

2. Let $p\colon E \to B$ be a covering map.

   (a) Show that if B is Hausdorff, E is Hausdorff.

   (b) Show that if B is compact and $p^{-1}(b)$ is finite for all $b \in B$., then E is compact.

   [Source: Fall 2019 205A Homework Set 5 Question 3]

3. Let X be a topological space and show that the following three conditions are equivalent:

   (a) Every map $S^1 \to X$ is null homotopic, i.e. homotopic to a constant map.

   (b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

   (c) $\pi_1(X, x_0)$ is trivial for all $x_0 \in X$.

   Deduce that a space X is simply-connected if and only if all maps $S^1 \to X$ are homotopic.
   [Source: Winter 2020 Math 205B Final Problem 1]

4. Let $p, q \geqslant 1$ be coprime integers. Define a space Z by attaching two 2-cells along their boundaries to S^1, the first by a map of degree p and the second to a map of degree q. Compute $H_n(Z)$ for all $n \geqslant 0$.
   [Source: Winter 2020 Math 205B Final Problem 3]

5. Let F_n the group on $n \geqslant 2$ generators.

   (a) Prove that any finitely generated subgroup of F_n is free.

   (b) Prove that any finitely generated normal subgroup of F_n is finite index.

6. Prove that $\mathbb{RP}^n$ is orientable if and only if n is odd.
   [Source: Spring 2020 205C Final Exam Problem 8]

7. Let M be a smooth manifold and $h\colon M \to N$ a homeomorphism. Prove that N admits a unique smooth structure such that h is a diffeomorphism.
   [Source: Spring 2020 205C Final Exam Problem 12]

8. Define $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\Phi(x, y) = x^2 - y^2$. Show that for all $c \neq 0$, $\Phi^{-1}(c)$ is a codimension 1 submanifold of $\mathbb{R}^2$. What about $\Phi^{-1}(0)$?
   [Source: Spring 2019 205C Final Exam Problem 11]